

# LARGE SCALE GEOMETRY OF COMMUTATOR SUBGROUPS

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ABSTRACT. Let  $G$  be a finitely presented group, and  $G'$  its commutator subgroup. Let  $C$  be the Cayley graph of  $G'$  with *all commutators in  $G$*  as generators. Then  $C$  is large scale simply connected. Furthermore, if  $G$  is a torsion-free nonelementary word-hyperbolic group,  $C$  is one-ended. Hence (in this case), the asymptotic dimension of  $C$  is at least 2.

## 1. INTRODUCTION

Let  $G$  be a group and let  $G' := [G, G]$  denote the commutator subgroup of  $G$ . The group  $G'$  has a canonical generating set  $S$ , which consists precisely of the set of commutators of pairs of elements in  $G$ . In other words,

$$S = \{[g, h] \text{ such that } g, h \in G\}$$

Let  $C_S(G')$  denote the Cayley graph of  $G'$  with respect to the generating set  $S$ . This graph can be given the structure of a (path) metric space in the usual way, where edges have length 1 by fiat.

By now it is standard to expect that the large scale geometry of a Cayley graph will reveal useful information about a group. However, one usually studies finitely generated groups  $G$  and the geometry of a Cayley graph  $C_T(G)$  associated to a finite generating set  $T$ . For typical infinite groups  $G$ , the set of commutators  $S$  will be infinite, and the Cayley graph  $C_S(G')$  will not be locally compact. This is a significant complication. Nevertheless,  $C_S(G')$  has several distinctive properties which invite careful study:

- (1) The set of commutators of a group is *characteristic* (i.e. invariant under any automorphism of  $G$ ), and therefore the semi-direct product  $G' \rtimes \text{Aut}(G)$  acts on  $C_S(G')$  by isometries
- (2) The metric on  $G'$  inherited as a subspace of  $C_S(G')$  is both left- and right-invariant (unlike the typical Cayley graph, whose metric is merely left-invariant)
- (3) Bounded cohomology in  $G$  is reflected in the geometry of  $G'$ ; for instance, the translation length  $\tau(g)$  of an element  $g \in G'$  is the stable commutator length  $\text{scl}(g)$  of  $g$  in  $G$
- (4) Simplicial loops in  $C_S(G')$  through the origin correspond to (marked) homotopy classes of maps of closed surfaces to a  $K(G, 1)$

These properties are straightforward to establish; for details, see § 2.

This paper concerns the connectivity of  $C_S(G')$  in the large for various groups  $G$ . Recall that a *thickening*  $Y$  of a metric space  $X$  is an isometric inclusion  $X \rightarrow Y$  into a bigger metric space, such that the Hausdorff distance in  $Y$  between  $X$  and  $Y$  is finite. A metric space  $X$  is said to be *large scale  $k$ -connected* if for any thickening

$Y$  of  $X$  there is another thickening  $Z$  of  $Y$  which is  $k$ -connected (i.e.  $\pi_i(Z) = 0$  for  $i \leq k$ ; also see the Definitions in § 3). Our first main theorem, proved in § 3, concerns the large scale connectivity of  $C_S(G')$  where  $G$  is finitely presented:

**Theorem A.** *Let  $G$  be a finitely presented group. Then  $C_S(G')$  is large scale simply connected.*

As well as large scale connectivity, one can study connectivity *at infinity*. In § 4 we specialize to word-hyperbolic  $G$  and prove our second main theorem, concerning the connectivity of  $G'$  at infinity:

**Theorem B.** *Let  $G$  be a torsion-free nonelementary word-hyperbolic group. Then  $C_S(G')$  is one-ended; i.e. for any  $r > 0$  there is an  $R \geq r$  such that any two points in  $C_S(G')$  at distance at least  $R$  from  $\text{id}$  can be joined by a path which does not come closer than distance  $r$  to  $\text{id}$ .*

Combined with a theorem of Fujiwara-Whyte [7], Theorem A and Theorem B together imply that for  $G$  a torsion-free nonelementary word-hyperbolic group,  $C_S(G')$  has asymptotic dimension at least 2 (see § 5 for the definition of asymptotic dimension).

## 2. DEFINITIONS AND BASIC PROPERTIES

Throughout the rest of this paper,  $G$  will denote a group,  $G'$  will denote its commutator subgroup, and  $S$  will denote the set of (nonzero) commutators in  $G$ , thought of as a generating set for  $G'$ . Let  $C_S(G')$  denote the Cayley graph of  $G'$  with respect to the generating set  $S$ . As a graph,  $C_S(G')$  has one vertex for every element of  $G'$ , and two elements  $g, h \in G'$  are joined by an edge if and only if  $g^{-1}h \in S$ . Let  $d$  denote distance in  $C_S(G')$  restricted to  $G'$ .

**Definition 2.1.** Let  $g \in G'$ . The *commutator length* of  $g$ , denoted  $\text{cl}(g)$ , is the smallest number of commutators in  $G$  whose product is equal to  $g$ .

From the definition, it follows that  $\text{cl}(g) = d(\text{id}, g)$  and  $d(g, h) = \text{cl}(g^{-1}h)$  for  $g, h \in G'$ .

**Lemma 2.2.** *The group  $G' \rtimes \text{Aut}(G)$  acts on  $C_S(G')$  by isometries.*

*Proof.*  $\text{Aut}(G)$  acts as permutations of  $S$ , and therefore the natural action on  $G$  extends to  $C_S(G')$ . Further,  $G'$  acts on  $C_S(G')$  by left multiplication.  $\square$

**Lemma 2.3.** *The metric on  $C_S(G')$  restricted to  $G'$  is left- and right-invariant.*

*Proof.* Since the inverse of a commutator is a commutator, we have  $\text{cl}(g^{-1}h) = \text{cl}(h^{-1}g)$ . Since the conjugate of a commutator by any element is a commutator, we have  $\text{cl}(h^{-1}g) = \text{cl}(gh^{-1})$ . This completes the proof.  $\square$

**Definition 2.4.** Given a metric space  $X$  and an isometry  $h$  of  $X$ , the *translation length* of  $h$  on  $X$ , denoted  $\tau(h)$ , is defined by the formula

$$\tau(h) = \lim_{n \rightarrow \infty} \frac{d(p, h^n(p))}{n}$$

where  $p \in X$  is arbitrary.

By the triangle inequality, the limit does not depend on the choice of  $p$ .

For  $g \in G'$  acting on  $C_S(G')$  by left multiplication, we can take  $p = \text{id}$ . Then  $d(\text{id}, g^n(\text{id})) = \text{cl}(g^n)$ .

**Definition 2.5.** Let  $G$  be a group, and  $g \in G'$ . The *stable commutator length* of  $g$  is the limit

$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n}$$

Hence we have the following:

**Lemma 2.6.** *Let  $g \in G'$  act on  $C_S(G')$  by left multiplication. There is an equality  $\tau(g) = \text{scl}(g)$ .*

*Proof.* This is immediate from the definitions.  $\square$

Stable commutator length is related to two-dimensional (bounded) cohomology. For an introduction to stable commutator length, see [3]; for an introduction to bounded cohomology, see [10].

If  $X$  is a metric space, and  $g$  is an isometry of  $X$ , one can obtain lower bounds on  $\tau(g)$  by constructing a Lipschitz function on  $X$  which grows linearly on the orbit of a point under powers of  $g$ . One important class of Lipschitz functions on  $C_S(G')$  are *quasimorphisms*:

**Definition 2.7.** Let  $G$  be a group. A function  $\phi : G \rightarrow \mathbb{R}$  is a *quasimorphism* if there is a least positive real number  $D(\phi)$  called the *defect*, such that for all  $g, h \in G$  there is an inequality

$$|\phi(g) + \phi(h) - \phi(gh)| \leq D(\phi)$$

From the defining property of a quasimorphism,  $|\phi(\text{id})| \leq D(\phi)$  and therefore by repeated application of the triangle inequality, one can estimate

$$|\phi(f[g, h]) - \phi(f)| \leq 7D(\phi)$$

for any  $f, g, h \in G$ . In other words,

**Lemma 2.8.** *Let  $G$  be a group, and let  $\phi : G \rightarrow \mathbb{R}$  be a quasimorphism with defect  $D(\phi)$ . Then  $\phi$  restricted to  $G'$  is  $7D(\phi)$ -Lipschitz in the metric inherited from  $C_S(G')$ .*

Word-hyperbolic groups admit a rich family of quasimorphisms. We will exploit this fact in § 4.

### 3. LARGE SCALE SIMPLE CONNECTIVITY

The following definitions are taken from [9], pp. 23–24.

**Definition 3.1.** A *thickening*  $Y$  of a metric space  $X$  is an isometric inclusion  $X \rightarrow Y$  with the property that there is a constant  $C$  so that every point in  $Y$  is within distance  $C$  of some point in  $X$ .

**Definition 3.2.** A metric space  $X$  is *large scale  $k$ -connected* if for every thickening  $X \subset Y$  there is a thickening  $Y \subset Z$  which is  $k$ -connected in the usual sense; i.e.  $Z$  is path-connected, and  $\pi_i(Z) = 0$  for  $i \leq k$ .

For  $G$  a finitely generated group with generating set  $T$ , Gromov outlines a proof ([9], 1.C<sub>2</sub>) that the Cayley graph  $C_T(G)$  is large scale 1-connected if and only if  $G$  is finitely presented, and  $C_T(G)$  is large scale  $k$ -connected if and only if there exists a proper simplicial action of  $G$  on a  $(k+1)$ -dimensional  $k$ -connected simplicial complex  $X$  with compact quotient  $X/G$ .

For  $T$  an infinite generating set, large scale simple connectivity is equivalent to the assertion that  $G$  admits a presentation  $G = \langle T \mid R \rangle$  where all elements in  $R$  have *uniformly bounded length* as words in  $T$ ; i.e. all relations in  $G$  are consequences of relations of bounded length.

To show that  $C_S(G')$  is large scale 1-connected, it suffices to show that there is a constant  $K$  so that for every simplicial loop  $\gamma$  in  $C_S(G')$  there are a sequence of loops  $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n$  where  $\gamma_n$  is the trivial loop, and each  $\gamma_i$  is obtained from  $\gamma_{i-1}$  by cutting out a subpath  $\sigma_{i-1} \subset \gamma_{i-1}$  and replacing it by a subpath  $\sigma_i \subset \gamma_i$  with the same endpoints, so that  $|\sigma_{i-1}| + |\sigma_i| \leq K$ .

More generally, we call the operation of cutting out a subpath  $\sigma$  and replacing it by a subpath  $\sigma'$  with the same endpoints where  $|\sigma| + |\sigma'| \leq K$  a  $K$ -move.

**Definition 3.3.** Two loops  $\gamma$  and  $\gamma'$  are  $K$ -equivalent if there is a finite sequence of  $K$ -moves which begins at  $\gamma$ , and ends at  $\gamma'$ .

$K$ -equivalence is (as the name suggests) an equivalence relation. The statement that  $C_S(G')$  is large scale 1-connected is equivalent to the statement that there is a constant  $K$  such that every two loops in  $C_S(G')$  are  $K$ -equivalent.

First we establish large scale simple connectivity in the case of a free group.

**Lemma 3.4.** *Let  $F$  be a finitely generated free group. Then  $C_S(F')$  is large scale simply connected.*

*Proof.* Let  $\gamma$  be a loop in  $C_S(F')$ . After acting on  $\gamma$  by left translation, we may assume that  $\gamma$  passes through  $\text{id}$ , so we may think of  $\gamma$  as a simplicial path in  $C_S(F')$  which starts and ends at  $\text{id}$ . If  $s_i \in S$  corresponds to the  $i$ th segment of  $\gamma$ , we obtain an expression

$$s_1 s_2 \cdots s_n = \text{id}$$

in  $F$ , where each  $s_i$  is a commutator. For each  $i$ , let  $a_i, b_i \in F$  be elements with  $[a_i, b_i] = s_i$  (note that  $a_i, b_i$  with this property are not necessarily unique). Let  $\Sigma$  be a surface of genus  $n$ , and let  $\alpha_i, \beta_i$  for  $i \leq n$  be a standard basis for  $\pi_1(\Sigma)$ ; see Figure 1.

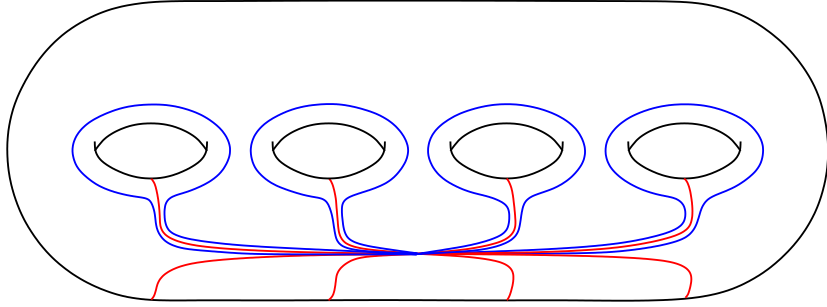


FIGURE 1. A standard basis for  $\pi_1(\Sigma)$  where  $\Sigma$  has genus 4. The  $\alpha_i$  curves are in red, and the  $\beta_i$  curves are in blue.

Let  $X$  be a wedge of circles corresponding to free generators for  $F$ , so that  $\pi_1(X) = F$ . We can construct a basepoint preserving map  $f : \Sigma \rightarrow X$  with  $f_*(\alpha_i) = a_i$  and  $f_*(\beta_i) = b_i$  for each  $i$ . Since  $X$  is a  $K(F, 1)$ , the homotopy class of  $f$  is uniquely determined by the  $a_i, b_i$ . Informally, we could say that loops in

$C_S(F')$  correspond to based homotopy classes of maps of marked oriented surfaces into  $X$  (up to the ambiguity indicated above).

Let  $\phi$  be a (basepoint preserving) self-homeomorphism of  $\Sigma$ . The map  $f \circ \phi : \Sigma \rightarrow X$  determines a new loop in  $C_S(F')$  (also passing through  $\text{id}$ ) which we denote  $\phi_*(\gamma)$  (despite the notation, this image does not depend only on  $\gamma$ , but on the choice of elements  $a_i, b_i$  as above).

**Sublemma 3.5.** *There is a universal constant  $K$  independent of  $\gamma$  or of  $\phi$  (or even of  $F$ ) so that after composing  $\phi$  by an inner automorphism of  $\pi_1(\Sigma)$  if necessary,  $\gamma$  and  $\phi_*(\gamma)$  as above are  $K$ -equivalent.*

*Proof.* Suppose we can express  $\phi$  as a product of (basepoint preserving) automorphisms

$$\phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$$

such that if  $\alpha_i^j, \beta_i^j$  denote the images of  $\alpha_i, \beta_i$  under  $\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1$ , then  $\phi_{j+1}$  fixes all but  $K$  consecutive pairs  $\alpha_i^j, \beta_i^j$  up to (basepoint preserving) homotopy. Let  $s_i^j = [f_*\alpha_i^j, f_*\beta_i^j]$ , and let  $\gamma^j$  be the loop in  $C_S(F')$  corresponding to the identity  $s_1^j s_2^j \cdots s_n^j = \text{id}$  in  $F$ .

For each  $j$ , let  $\text{supp}_{j+1}$  denote the *support* of  $\phi_{j+1}$ ; i.e. the set of indices  $i$  such that  $\phi_{j+1}(\alpha_i^j) \neq \alpha_i^j$  or  $\phi_{j+1}(\beta_i^j) \neq \beta_i^j$ . By hypothesis,  $\text{supp}_{j+1}$  consists of at most  $K$  indices for each  $j$ .

Because it is just the marking on  $\Sigma$  which has been changed and not the map  $f$ , if  $k \leq i \leq k + K - 1$  is a maximal consecutive string of indices in  $\text{supp}_{j+1}$ , then there is an equality of products

$$s_k^j s_{k+1}^j \cdots s_{k+K-1}^j = s_k^{j+1} s_{k+1}^{j+1} \cdots s_{k+K-1}^{j+1}$$

as elements of  $F$ . This can be seen geometrically as follows. The expression on the left is the image under  $f_*$  of an element represented by a certain embedded based loop in  $\Sigma$ , while the expression on the right is its image under  $f_* \circ \phi_{j+1}$ . The automorphism  $\phi_{j+1}$  is represented by a homeomorphism of  $\Sigma$  whose support is contained in regions bounded by such loops. Hence the expressions are equal. It follows that  $\gamma^j$  and  $\gamma^{j+1}$  are  $2K$ -equivalent.

So to prove the Sublemma it suffices to show that any automorphism of  $S$  can be expressed (up to inner automorphism) as a product of automorphisms  $\phi_i$  with the property above.

The hypothesis that we may compose  $\phi$  by an inner automorphism means that we need only consider the image of  $\phi$  in the mapping class group of  $\Sigma$ . It is well-known since Dehn [5] that the mapping class group of a closed oriented surface  $\Sigma$  of genus  $g$  is generated by twists in a finite standard set of curves, each of which intersects at most two of the  $\alpha_i, \beta_i$  essentially; see Figure 2.

So write  $\phi = \tau_1 \tau_2 \cdots \tau_m$  where the  $\tau_i$  are all standard generators. Now define

$$\phi_j = \tau_1 \tau_2 \cdots \tau_{j-1} \tau_j \tau_{j-1}^{-1} \cdots \tau_1^{-1}$$

We have

$$\phi_j \phi_{j-1} \cdots \phi_1 = \tau_1 \tau_2 \cdots \tau_j$$

Moreover, each  $\phi_j$  is a Dehn twist in a curve which is the image of a standard curve under  $\phi_{j-1} \cdots \phi_1$ , and therefore intersects  $\alpha_i^{j-1}, \beta_i^{j-1}$  essentially for at most 2 (consecutive) indices  $i$ . This completes the proof of the Sublemma (and shows, in fact, that we can take  $K = 4$ ).  $\square$

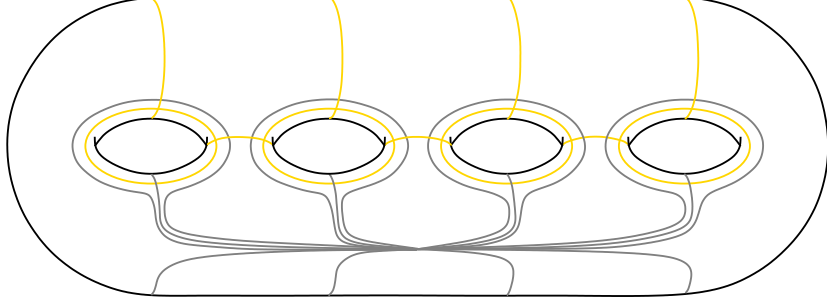


FIGURE 2. A standard set of  $3g - 1$  simple curves, in yellow. Dehn twists in these curves generate the mapping class group of  $\Sigma$ .

We now complete the proof of the Lemma. As observed by Stallings (see e.g. [14]), a nontrivial map  $f : \Sigma \rightarrow X$  from a closed, oriented surface to a wedge of circles factors (up to homotopy) through a *pinch* in the following sense. Make  $f$  transverse to some edge  $e$  of  $X$ , and look at the preimage  $\Gamma$  of a regular value of  $f$  in  $e$ . After homotoping inessential loops of  $\Gamma$  off  $e$ , we may assume that for some edge  $e$  and some regular value, the preimage  $\Gamma$  contains an embedded essential loop  $\delta$ .

There are two cases to consider. In the first case,  $\delta$  is nonseparating. In this case, let  $\phi$  be an automorphism which takes  $\alpha_1$  to the free homotopy class of  $\delta$ . Then  $\gamma$  and  $\phi_*(\gamma)$  are  $K$ -equivalent by the Sublemma. However, since  $f(\delta)$  is homotopically trivial in  $X$ , there is an identity  $[\phi_*\alpha_1, \phi_*\beta_1] = \text{id}$  and therefore  $\phi_*(\gamma)$  has length 1 shorter than  $\gamma$ .

In the second case,  $\phi$  is separating, and we can let  $\phi$  be an automorphism which takes the free homotopy class of  $[\alpha_1, \beta_1] \cdots [\alpha_j, \beta_j]$  to  $\delta$ . Again, by the Sublemma,  $\gamma$  and  $\phi_*(\gamma)$  are  $K$ -equivalent. But now  $\phi_*(\gamma)$  contains a subarc of length  $j$  with both endpoints at  $\text{id}$ , so we may write it as a product of two loops at  $\text{id}$ , each of length shorter than that of  $\gamma$ .

By induction,  $\gamma$  is  $K$ -equivalent to the trivial loop, and we are done.  $\square$

We are now in a position to prove our first main theorem.

**Theorem A.** *Let  $G$  be a finitely presented group. Then  $C_S(G')$  is large scale simply connected.*

*Proof.* Let  $W$  be a smooth 4-manifold (with boundary) satisfying  $\pi_1(W) = G$ . If  $G = \langle T \mid R \rangle$  is a finite presentation, we can build  $W$  as a handlebody, with one 0-handle, one 1-handle for every generator in  $T$ , and one 2-handle for every relation in  $R$ . If  $r_i \in R$  is a relation, let  $D_i$  be the cocore of the corresponding 2-handle, so that  $D_i$  is a properly embedded disk in  $W$ . Let  $V \subset W$  be the union of the 0-handle and the 1-handles. Topologically,  $V$  is homotopy equivalent to a wedge of circles. By the definition of cocores, the complement of  $\cup_i D_i$  in  $W$  deformation retracts to  $V$ . See e.g. [12], Chapter 1 for an introduction to handle decompositions of 4-manifolds.

Given  $\gamma$  a loop in  $C_S(G')$ , translate it by left multiplication so that it passes through  $\text{id}$ . As before, let  $\Sigma$  be a closed oriented marked surface, and  $f : \Sigma \rightarrow W$  a map representing  $\gamma$ .

Since  $G$  is finitely presented,  $H_2(G; \mathbb{Z})$  is finitely generated. Choose finitely many closed oriented surfaces  $S_1, \dots, S_r$  in  $W$  which generate  $H_2(G; \mathbb{Z})$ . Let  $K'$  be the supremum of the genus of the  $S_i$ . We can choose a basepoint on each  $S_i$ , and maps to  $W$  which are basepoint preserving. By tubing  $\Sigma$  repeatedly to copies of the  $S_i$  with either orientation, we obtain a new surface and map  $f' : \Sigma' \rightarrow W$  representing a loop  $\gamma'$  such that  $f'(\Sigma')$  is null-homologous in  $W$ , and  $\gamma'$  is  $K'$ -equivalent to  $\gamma$  (note that  $K'$  depends on  $G$  but not on  $\gamma$ ).

Put  $f'$  in general position with respect to the  $D_i$  by a homotopy. Since  $f'(\Sigma')$  is null-homologous, for each proper disk  $D_i$ , the signed intersection number vanishes:  $D_i \cap f'(\Sigma') = 0$ . Hence  $f'(\Sigma) \cap D_i = P_i$  is a finite, even number of points which can be partitioned into two sets of equal size corresponding to the local intersection number of  $f'(\Sigma')$  with  $D_i$  at  $p \in P_i$ .

Let  $p, q \in P_i$  have opposite signs, and let  $\mu$  be an embedded path in  $D_i$  from  $f'(p)$  to  $f'(q)$ . Identifying  $p$  and  $q$  implicitly with their preimages in  $\Sigma'$ , let  $\alpha$  and  $\beta$  be arcs in  $\Sigma'$  from the basepoint to  $(f')^{-1}p$  and  $(f')^{-1}q$ . Since  $\mu$  is contractible, there is a neighborhood of  $\mu$  in  $D_i$  on which the normal bundle is trivializable. Hence, since  $f'(\Sigma')$  and  $D_i$  are transverse, we can find a neighborhood  $U$  of  $\mu$  in  $W$  disjoint from the other  $D_j$ , and co-ordinates on  $U$  satisfying

- (1)  $D_i \cap U$  is the plane  $(x, y, 0, 0)$
- (2)  $\mu \cap U$  is the interval  $(t, 0, 0, 0)$  for  $t \in [0, 1]$
- (3)  $f'(\Sigma') \cap U$  is the union of the planes  $(0, 0, z, w)$  and  $(1, 0, z, w)$

Let  $A$  be the annulus consisting of points  $(t, 0, \cos(\theta), \sin(\theta))$  where  $t \in [0, 1]$ . Then  $A$  is disjoint from  $D_i$  and all the other  $D_j$ , and we can tube  $f'(\Sigma')$  with  $A$  to reduce the number of intersection points of  $f'(\Sigma')$  with  $\cup_i D_i$ , at the cost of raising the genus by 1. Technically, we remove the disks  $(f')^{-1}(0, 0, s \cos(\theta), s \sin(\theta))$  and  $(f')^{-1}(1, 0, s \cos(\theta), s \sin(\theta))$  for  $s \in [0, 1]$  from  $\Sigma'$ , and sew in a new annulus which we map homeomorphically to  $A$ . The result is  $f'' : \Sigma'' \rightarrow W$  with two fewer intersection points with  $\cup_i D_i$ . This has the effect of adding a new (trivial) edge to the start of  $\gamma'$ , which is the commutator of the elements represented by the core of  $A$  and the loop  $f'(\alpha) * \mu * f'(\beta)$ . Let  $\gamma''$  denote this resulting loop, and observe that  $\gamma''$  is 1-equivalent to  $\gamma'$ . After finitely many operations of this kind, we obtain  $f''' : \Sigma''' \rightarrow W$  corresponding to a loop  $\gamma'''$  which is  $\max(1, K')$ -equivalent to  $\gamma$ , such that  $f'''(\Sigma''')$  is disjoint from  $\cup_i D_i$ .

After composing with a deformation retraction, we may assume  $f'''$  maps  $\Sigma'''$  into  $V$ . Let  $F = \pi_1(V)$ , and let  $\rho : F \rightarrow G$  be the homomorphism induced by the inclusion  $V \rightarrow W$ . There is a loop  $\gamma^F$  in  $C_S(F')$  corresponding to  $f'''$  such that  $\rho_*(\gamma^F) = \gamma'''$  under the obvious simplicial map  $\rho_* : C_S(F') \rightarrow C_S(G')$ . By Lemma 3.4, the loop  $\gamma^F$  is  $K$ -equivalent to a trivial loop in  $C_S(F')$ . Pushing forward the sequence of intermediate loops by  $\rho_*$  shows that  $\gamma'''$  is  $K$ -equivalent to a trivial loop in  $C_S(G')$ . Since  $\gamma$  was arbitrary, we are done.  $\square$

*Remark 3.6.* A similar, though perhaps more combinatorial argument could be made working directly with 2-complexes in place of 4-manifolds.

In words, Theorem A says that for  $G$  a finitely presented group, all relations amongst the commutators of  $G$  are consequences of relations involving only boundedly many commutators.

The next example shows that the size of this bound depends on  $G$ :

*Example 3.7.* Let  $\Sigma$  be a closed surface of genus  $g$ , and  $G = \pi_1(\Sigma)$ . If  $\gamma$  is a loop in  $C_S(G)$  through the origin, and  $f : \Sigma' \rightarrow \Sigma$  is a corresponding map of a closed surface, then the homology class of  $\Sigma'$  is trivial unless the genus of  $\Sigma'$  is at least as big as that of  $\Sigma$ . Hence the loop in  $C_S(G)$  of length  $g$  corresponding to the relation in the “standard” presentation of  $\pi_1(\Sigma)$  is not  $K$ -equivalent to the trivial loop whenever  $K < g$ .

In light of Theorem A, it is natural to ask the following question:

**Question 3.8.** *Let  $G$  be a finitely presented group. Is  $C_S(G')$  large scale  $k$ -connected for all  $k$ ?*

*Remark 3.9.* Laurent Bartholdi has pointed out that for  $F$  a finitely generated free group, there is a confluent, Noetherian rewriting system for  $F'$ , with rules of bounded length, which puts every word in  $F'$  over generators  $S$  into normal form (with respect to a “standard” free generating set for  $F'$ ). By results of Groves ([11]) this should imply that  $C_S(F')$  is large scale  $k$ -connected for all  $k$ , but we have not verified this implication carefully. In any case, it gives another more algebraic proof of Lemma 3.4.

#### 4. WORD-HYPERBOLIC GROUPS

In this section we specialize to the class of *word-hyperbolic groups*. See [8] for more details.

**Definition 4.1.** A path metric space  $X$  is  $\delta$ -hyperbolic for some  $\delta \geq 0$  if for every geodesic triangle  $abc$ , and every point  $p$  on the edge  $ab$ , there is  $q \in ac \cup bc$  with  $d_X(p, q) \leq \delta$ . In other words, the  $\delta$  neighborhood of any two sides of a geodesic triangle contains the third side.

**Definition 4.2.** A group  $G$  is *word-hyperbolic* if there is a finite generating set  $T$  for  $G$  such that  $C_T(G)$  is  $\delta$ -hyperbolic as a path metric space, for some  $\delta$ .

*Example 4.3.* Finitely generated free groups are word-hyperbolic. The fundamental group of a closed surface with negative Euler characteristic is word-hyperbolic. Discrete cocompact groups of isometries of hyperbolic  $n$ -space are word-hyperbolic.

To rule out some trivial examples, one makes the following:

**Definition 4.4.** A word-hyperbolic group is *elementary* if it has a cyclic subgroup of finite index, and *nonelementary* otherwise.

The main theorem we prove in this section concerns the geometry of  $C_S(G')$  at infinity, where  $G$  is a nonelementary word-hyperbolic group. For the sake of brevity we restrict attention to torsion-free  $G$ , though this restriction is not logically necessary; see Remark 4.9.

**Theorem B.** *Let  $G$  be a torsion-free nonelementary word-hyperbolic group. Then  $C_S(G')$  is one-ended; i.e. for any  $r > 0$  there is an  $R \geq r$  such that any two points in  $C_S(G')$  at distance at least  $R$  from  $\text{id}$  can be joined by a path which does not come closer than distance  $r$  to  $\text{id}$ .*

We will estimate distance to  $\text{id}$  in  $C_S(G')$  using quasimorphisms, as indicated in § 2. Hyperbolic groups admit a rich family of quasimorphisms. Of particular



interest to us are the *Epstein-Fujiwara counting quasimorphisms*, introduced in [6], generalizing a construction due to Brooks [2] for free groups.

Fix a word-hyperbolic group  $G$  and a finite generating set  $T$ . Let  $C_T(G)$  denote the Cayley graph of  $G$  with respect to  $T$ . Let  $\sigma$  be an oriented simplicial path in  $C_T(G)$ . A *copy* of  $\sigma$  is a translate  $g \cdot \sigma$  for some  $g \in G$ . If  $\gamma$  is an oriented simplicial path in  $C_T(G)$ , let  $|\gamma|_\sigma$  denote the maximal number of disjoint copies of  $\sigma$  contained in  $\gamma$ . For  $g \in G$ , define

$$c_\sigma(g) = d(\text{id}, g) - \inf_\gamma (\text{length}(\gamma) - |\gamma|_\sigma)$$

where the infimum is taken over all directed paths  $\gamma$  in  $C_T(G)$  from  $\text{id}$  to  $g$ , and  $d(\cdot, \cdot)$  denotes distance in  $C_T(G)$ .

**Definition 4.5** (Epstein-Fujiwara). A *counting quasimorphism* on  $G$  is a function of the form

$$h_\sigma(g) := c_\sigma(g) - c_{\sigma^{-1}}(g)$$

where  $\sigma^{-1}$  denotes the same simplicial path as  $\sigma$  with the opposite orientation.

Since  $|\gamma|_\sigma$  takes discrete values, the infimum is realized in the definition of  $c_\sigma$ . A path  $\gamma$  for which

$$c_\sigma(g) = d(\text{id}, g) - \text{length}(\gamma) + |\gamma|_\sigma$$

is called a *realizing path* for  $g$ . Realizing paths exist, and satisfy the following geometric property:

**Lemma 4.6** (Epstein-Fujiwara, [6] Prop. 2.2). *Any realizing path for  $g$  is a  $K, \epsilon$ -quasigeodesic in  $C_T(G)$ , where*

$$K = \frac{\text{length}(\sigma)}{\text{length}(\sigma) - 1}, \quad \text{and} \quad \epsilon = \frac{2 \cdot \text{length}(\sigma)}{\text{length}(\sigma) - 1}$$

Moreover,

**Lemma 4.7** (Epstein-Fujiwara, [6] Prop. 2.13). *Let  $\sigma$  be a path in  $C_T(G)$  of length at least 2. Then there is a constant  $K(\delta)$  (where  $T$  is such that  $C_T(G)$  is  $\delta$ -hyperbolic as a metric space) such that  $D(h_\sigma) \leq K(\delta)$ .*

Counting quasimorphisms are very versatile, as the following lemma shows:

**Lemma 4.8.** *Let  $G$  be a torsion-free, nonelementary word-hyperbolic group. Let  $g_i$  be a finite collection of elements of  $G$ . There is a commutator  $s \in G'$  and a quasimorphism  $\phi$  on  $G$  with the following properties:*

- (1)  $|\phi(g_i)| = 0$  for all  $i$
- (2)  $|\phi(s^n) - n| \leq K_1$  for all  $n$ , where  $K_1$  is a constant which depends only on  $G$ .
- (3)  $D(\phi) \leq K_2$  where  $K_2$  is a constant which depends only on  $G$

*Proof.* Fix a finite generating set  $T$  so that  $C_T(G)$  is  $\delta$ -hyperbolic. There is a constant  $N$  such that for any nonzero  $g \in G$ , the power  $g^N$  fixes an axis  $L_g$  ([8]). Since  $G$  is nonelementary, it contains quasigeodesically embedded copies of free groups, of any fixed rank. So we can find a commutator  $s$  whose translation length (in  $C_T(G)$ ) is as big as desired. In particular, given  $g_1, \dots, g_j$  we choose  $s$  with  $\tau(s) \gg \tau(g_i)$  for all  $i$ . Let  $L$  be a geodesic axis for  $s^N$ , and let  $\sigma$  be a fundamental domain for the action of  $s^N$  on  $L$ . Since  $|\sigma| = N\tau(s) \gg \tau(g_i)$ , Lemma 4.6 implies

that there are no copies of  $\sigma$  or  $\sigma^{-1}$  in a realizing path for any  $g_i$ . Hence  $h_\sigma(g_i) = 0$  for all  $i$ . By Lemma 4.7,  $D(h_\sigma) \leq K(\delta)$ . It remains to estimate  $h_\sigma(s^n)$ .

In fact, the argument of [4] Theorem A' (which establishes explicitly an estimate that is implicit in [6]) shows that for  $N$  sufficiently large (depending only on  $G$  and not on  $s$ ) no copies of  $\sigma^{-1}$  are contained in any realizing path for  $s^n$  with  $n$  positive, and therefore  $|h_\sigma(s^n) - \lfloor n/N \rfloor|$  is bounded by a constant depending only on  $G$ . The quasimorphism  $\phi = N \cdot h_\sigma$  has the desired properties.  $\square$

*Remark 4.9.* The hypothesis that  $G$  is torsion-free is included only to ensure that  $s$  is not conjugate to  $s^{-1}$ . It is possible to remove this hypothesis by taking slightly more care in the definition of  $s$ , using the methods of the proof of Proposition 2 from [1]. We are grateful to the referee for pointing this out.

We now give the proof of Theorem B:

*Proof.* Let  $g, h \in G'$  have commutator length at least  $R$ . Let  $g = s_1 s_2 \cdots s_n$  and  $h = t_1 t_2 \cdots t_m$  where  $n, m \geq R$  are equal to the commutator lengths of  $g$  and  $h$  respectively, and each  $s_i, t_i$  is a commutator in  $G$ . Let  $s$  be a commutator with the properties described in Lemma 4.8 with respect to the elements  $g, h$ ; that is, we want  $s$  for which there is a quasimorphism  $\phi$  with  $\phi(g) = \phi(h) = 0$ , with  $|\phi(s^n) - n| \leq K_1$  for all  $n$ , and with  $D(\phi) \leq K_2$ . Let  $N \gg R$  be very large. We build a path in  $C_S(G')$  from  $g$  to  $h$  out of four segments, none of which come too close to id.

The first segment is

$$g, gs, gs^2, gs^3, \dots, gs^N$$

Since  $s$  is a commutator,  $d(gs^i, \text{id}) \geq R - i$  for any  $i$ . On the other hand,

$$\phi(gs^i) \geq \phi(g) + \phi(s^i) - D(\phi) \geq i - K_2 - K_1$$

where  $K_1, K_2$  are as in Lemma 4.8 (and do not depend on  $g, h, s$ ). From Lemma 2.8 we can estimate

$$d(gs^i, \text{id}) \geq \frac{\phi(gs^i)}{7D(\phi)} \geq \frac{i - K_2 - K_1}{7K_2}$$

Hence  $d(gs^i, \text{id}) \geq R/14K_2 - (K_1 + K_2)/7K_2$  for all  $i$ , so providing  $R \gg K_1, K_2$ , the path  $gs^i$  never gets too close to id.

The second segment is

$$gs^N = s_1 s_2 \cdots s_n s^N, s_2 \cdots s_n s^N, \dots, s^N$$

Note that consecutive elements in this segment are distance 1 apart in  $C_S(G')$ , by Lemma 2.3. Since  $d(gs^N, \text{id}) \geq (N - K_2 - K_1)/7K_2 \gg R$  for  $N$  sufficiently large, we have

$$d(s_i \cdots s_n s^N, \text{id}) \gg R$$

for all  $i$ .

The third segment is

$$s^N, t_m s^N, t_{m-1} t_m s^N, \dots, t_1 t_2 \cdots t_m s^N = h s^N$$

and the fourth is

$$h s^N, h s^{N-1}, \dots, h s, h$$

For the same reason as above, neither of these segments gets too close to id. This completes the proof of the theorem, taking  $r = R/14K_2 - (K_1 + K_2)/7K_2$ .  $\square$

## 5. ASYMPTOTIC DIMENSION

The main point of this section is to make the observation that  $G'$  for  $G$  as above is not a quasi-tree, and to restate this observation in terms of asymptotic dimension. We think it is worth making this restatement explicitly. The notion of asymptotic dimension is introduced in [9], p. 32.

**Definition 5.1.** Let  $X$  be a metric space, and  $X = \cup_i U_i$  a covering by subsets. For given  $D \geq 0$ , the  $D$ -multiplicity of the covering is at most  $n$  if for any  $x \in X$ , the closed  $D$ -ball centered at  $x$  intersects at most  $n$  of the  $U_i$ .

A metric space  $X$  has *asymptotic dimension at most  $n$*  if for every  $D \geq 0$  there is a covering  $X = \cup_i U_i$  for which the diameters of the  $U_i$  are uniformly bounded, and the  $D$ -multiplicity of the covering is at most  $n + 1$ . The least such  $n$  is the *asymptotic dimension* of  $X$ , and we write

$$\text{as dim}(X) = n$$

If  $X$  is a metric space, we say  $H_1(X)$  is *uniformly generated* if there is a constant  $L$  such that  $H_1(X)$  is generated by loops of length at most  $L$ . It is clear that if  $X$  is large scale 1-connected, then  $H_1(X)$  is uniformly generated. Fujiwara-Whyte [7] prove the following theorem:

**Theorem 5.2** (Fujiwara-Whyte, [7], Thm. 0.1). *Let  $X$  be a geodesic metric space with  $H_1(X)$  uniformly generated.  $X$  has  $\text{as dim}(X) = 1$  if and only if  $X$  is quasi-isometric to an unbounded tree.*

A group whose Cayley graph is quasi-isometric to an unbounded tree has more than one end (see e.g. Manning [13], especially § 2.1 and § 2.2). Hence Theorem A and Theorem B together imply the following:

**Corollary 5.3.** *Let  $G$  be a nonelementary torsion-free word-hyperbolic group. Then*

$$\text{as dim}(C_S(G')) \geq 2$$

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